Overlap-Save and Overlap-Add Filters: Optimal Design and Comparison

Ali Daher, El Houssaïn Baghious, Gilles Burel, Senior Member, IEEE, and Emanuel Radoi

Abstract—Overlap-save (OLS) and overlap-add (OLA) are two techniques widely used in digital filtering. In traditional OLS and OLA implementations, the system is compelled to be time-invariant and conventional filter synthesis techniques are used for designing the block filter. In this paper, based on the OLS and the OLA structures, we develop a fast algorithm for designing the optimal OLS and OLA block filters using a quadratic criterion. Comparing OLA to OLS optimal design, we demonstrate that, as in classical design approaches, they show no difference when the filters are time-invariant. However, when aliasing is not zero, although the global aliasing is the same, its components with respect to frequency are different. This conclusion is supported by simulation results, and a comparison between the optimal approach and some other standard approaches is also provided.

Index Terms—Aliasing, block digital filters, circulant matrices, digital filter design, overlap-save/add, time-invariant systems.

I. INTRODUCTION

FILTERING in real time a long-duration signal is a familiar signal processing problem in digital filtering systems. For long filter length, transform-based block digital filtering is a powerful tool to reduce the computational complexity. The best known and most widely used approaches are overlap-save (OLS) and overlap-add (OLA) [10, p. 558] where long signals are broken into smaller segments for easier processing by blocks using the fast Fourier transform (FFT) or other fast transforms. When the block sizes and the transform, the discrete Fourier transform (DFT) in our case, are chosen, the design of a the OLS and OLA block digital filters (BDFs) consists of optimizing the filter’s parameters in order to obtain a frequency response close to the desired one. A generalization of the block processing filtering and the conditions for its shift-invariant operation have been developed in [2].

In traditional OLS and OLA implementation, the system is generally compelled to be time-invariant i.e., aliasing error is null. This ensures equality between the linear convolution (direct filtering) and the circular convolution (block filtering). Traditional synthesis methods, such as the Hamming window design, the least square method design or others are often used for designing the block filter. For applications for which aliasing is tolerated, a sampling frequency approach is straightforward: direct sampling of the desired frequency response whose samples can be interpreted as weights applied to the frequency representation of the input block.

Based on the OLS implementation structure, a simple efficient matrix-oriented approach to optimize the BDF coefficients has been proposed in [1]. The criterion defined for the optimal BDF design consists in minimizing the global quadratic distortion between the desired filtering and the block filtering. Although the aliasing distortion is not zero, the global distortion error associated to this optimal approach is lower than that associated to the other traditional approaches.

In this paper, based on the work cited above [1], we will develop the optimal block filter design for OLA implementation. After that, we will derive the algorithm for designing the optimal time-invariant block filters. Comparing the optimal OLS design to the optimal OLA design, we will demonstrate that, as in traditional time-invariant design methods, they show no difference when the system is time-invariant. However, when the filter is time-variant, the aliasing components will not have the same distribution with respect to frequency. At the end, we will compare the optimal OLS/OLA filter design to the traditional OLS/OLA filter design techniques and we will show simulation results.

The paper is organized as follows. In Section II, the mathematical model of the desired filtering is described. In Section III, the optimal OLS design approach is presented. In Section IV, the optimal OLA design approach is developed. In Section V, the design algorithm for optimal time-invariant block filter is described. OLS and OLA implementations are compared in Section VI. Experimental results are shown in Section VII. Finally, some conclusions are drawn in Section VIII.

Throughout the paper, the first row/column index is index is 0. Then, for an \( m \times n \) matrix \( A \), we will use the following notations:

\[
\begin{align*}
A^T & \quad \text{the } n \times m \text{ transpose matrix of } A; \\
A^* & \quad \text{the conjugate transpose matrix of } A; \\
A(:,j) & \quad \text{the } m \times 1 \text{ vector formed by the elements of column } j \text{ of matrix } A; \\
A(i,:)^T & \quad \text{the } n \times 1 \text{ vector formed by the elements of row } i \text{ of matrix } A; \\
A(i,j) & \quad \text{the element corresponding to the intersection of row } i \text{ and column } j \text{ of matrix } A; \\
A(i_1 : i_2,j) & \quad \text{the matrix } A \text{ restricted to rows } i_1 \text{ to } i_2; \\
A(:,j_1 : j_2) & \quad \text{the matrix } A \text{ restricted to columns } j_1 \text{ to } j_2; \\
\mid A(i,j) \mid & \quad \text{the modulus of the element } A(i,j).
\end{align*}
\]
We will note \( F_K \) as the \( K \times K \) matrix corresponding to the \( K \)-point DFT and \( F_K^{-1} \) as its inverse corresponding to the \( K \)-point inverse DFT:

\[
\begin{align*}
F_K(m, n) &= e^{-j\frac{2\pi}{K} mn} = \alpha_K^{mn} (m, n : 0 \rightarrow K - 1) \\
F_K^{-1}(m, n) &= \frac{1}{K} e^{j\frac{2\pi}{K} mn} \\
&= \frac{1}{K} \alpha_K^{mn} (m, n : 0 \rightarrow K - 1)
\end{align*}
\]

where

\[
\alpha_K = e^{j\frac{2\pi}{K}}.
\]

We also define the following.

- \( 0_{m \times n} \) the \( m \times n \) zero matrix;
- \( I_m \) the \( m \times m \) identity matrix;
- \( S \) the \( L \times M \) selection matrix, a binary matrix which selects the \( L \) middle coefficients of an \( M = L + 2d \) length vector:

\[
S = [0_{L \times d} I_L 0_{L \times d}];
\]

- \( P_M \) the \( M \times M \) circulant permutation matrix of the cyclic right shift operator and \( P_M^{-1} \) as its inverse:

\[
\begin{align*}
P_M &= \begin{bmatrix}
0_{M-1 \times M-1} & I_1 \\
I_{M-1} & 0_{M-1 \times 1}
\end{bmatrix} \\
P_M^{-1} &= \begin{bmatrix}
0_{M-1 \times 1} & I_{M-1} \\
I_1 & 0_{1 \times M-1}
\end{bmatrix};
\end{align*}
\]

\( \langle \rangle_d \) as the modulo \( q \) operation.

In the paper, we will use the properties of circulant matrices. An \( M \times M \) circulant matrix \( C \) is formed from an \( M \) length vector by cyclically permuting the entries:

\[
C = \begin{bmatrix}
\alpha_0 & c_1 & \cdots & \cdots & c_{M-1} \\
c_{M-1} & \alpha_0 & c_1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
c_1 & \cdots & \cdots & \cdots & \alpha_0
\end{bmatrix}
\]

The elements of a circulant matrix \( C \) are related by

\[
C(i, j) = C(i + m)_{M}, (j + m)_{M}) = c(j - i)_{M} (3)
\]

\( i, j : 0 \rightarrow M - 1 \). In vector-matrix form, the relation between the rows (respectively the columns) of a circulant matrix \( C \) are described by

\[
\begin{align*}
C(n,:)^T &= P_M C(n-1,:)^T = (P_M)^n C(0,:)^T \\
C(:,n) &= P_M C(:,n-1) = (P_M)^n C(:,0)
\end{align*}
\]

\( n = 0, 1, \ldots, M - 1 \). Circulant matrices have many mathematical properties which are very useful in digital processing, communications and information theory. The most interesting property is matrix diagonalization [3]. An \( M \)-circulant matrix matrix \( C \) can be decomposed as

\[
C = F_M^{-1} D F_M
\]

where \( F_M \) is the \( M \times M \) matrix corresponding to the \( M \)-point DFT and \( D \) is an \( M \times M \) diagonal matrix, the diagonal of which (eigenvalues of the matrix \( C \)) is obtained by

\[
diag(D) = F_M C(:,0) = M F_M^{-1} C(:,0)^T.
\]

### II. Desired Filtering

The frequency response of any digital signal with frequency sampling \( F_s \) is periodic with period \( F_s \). In practice, the output spectrum of the filter is always analyzed with a limited frequency resolution \( \theta \). We can suppose, without any practical inconvenience, that \( \theta \) is chosen such that \( \theta = F_s / K \), where \( K \) is an integer (therefore, choosing \( K \) is equivalent to choosing the frequency resolution \( \theta \) and vice versa).

Since there is a bijection between the set of frequency responses with period \( F_s \) and frequency resolution \( \theta = F_s / K \), and the set of periodical digital signals with sampling frequency \( F_s \) and period \( K F_s \) (this bijection is given by the \( K \)-point DFT and its inverse), we can, without loss of generality, explain our approach by considering \( K \)-point signals. This does not mean, of course, that our approach is restricted to \( K \)-length signals, but this makes the calculations much clearer.

In the following, the input signal will be denoted \( x \) and its \( K \)-point DFT will be denoted \( \overline{x} \).

The desired output signal \( y_d \) and its DFT \( \overline{y_d} \) are then given by

\[
\overline{y_d} = \overline{H_d x} \quad \leftrightarrow \quad y_d = H_d x.
\]

\( \overline{H_d} \) is the diagonal matrix with

\[
\overline{H_d} (k, k) = f(k) (k = 0, 1, \ldots K - 1)
\]

where \( f(k) \) is the \( K \)-length desired frequency response vector and \( H_d \) is the \( K \times K \) matrix defined by

\[
H_d = F_K^{-1} \overline{H_d} F_K.
\]

### III. Overlap-Save

#### A. Overlap-Save Scheme

Block digital filtering using the OLS scheme is illustrated in Fig. 1. The input signal is divided into overlapping blocks \( x_i \) of \( M \) samples. The amount of overlapping is \( M - L \). Then, for each block, the following computations are performed.

- The block \( x_i \) is transformed through an \( M \)-point DFT.
- The transformed block vector coefficients are multiplied term by term by an \( M \)-length vector \( g_s \). In traditional methods, this vector \( g_s \) corresponds to the \( M \)-point DFT of the filter impulse response \( h \) designed by one of the standard filter design methods.
- An \( M \)-point inverse DFT (IDFT) is performed.
— Only the $L$ central points of the resulting block are kept. Then, each $M$ length input block $x_i$ provides an $L$-length output block $y_i$ ($L \leq M$ and $M - L$ is even to preserve symmetry).
— The concatenation of the output blocks forms the filtered output signal.

An $M$-length DFT/IDFT with complex inputs via the split-radix FFT (SRFFT) algorithm requires $M(\log_2 M - 3)/2 + 4$ real multiplications and $3M(\log_2 M - 1)/2 + 4$ real additions [5, p. 57–60], that is, $O(M \log_2 M)$ real multiplications and $O(3M \log_2 M)$ real additions. Generally, a multiplication requires more execution time than an addition, and then the computational complexity of an algorithm can be associated to the number of real multiplications involved in the operation. We consider that each complex multiplication is equivalent to three real multiplications according to Golub’s method. Hence, the number of real multiplications per output sample (rmpos) in OLS block filtering (DFT + term by term multiplication + IDFT) is given by $\mu \approx (M(2\log_2 M + 3))/L$ (rmpos).

B. Optimal Overlap-Save Filter Design

For the same cost filtering and the same structure of OLS shown above, the filter design can be optimized so that the filter distortion error will be lower than that obtained with classical filter design. The optimal approach developed in [1] takes into account both time-invariant and aliasing components, that is the BDF is not compelled to be time-invariant. It is optimal with respect to a quadratic criterion as described below.

By referring to Fig. 1, the input block $x_i$ and the output block $y_i$ are linearly related by

$$y_i = A_s x_i,$$  (11)

$A_s$ is the $L \times M$ matrix given by

$$A_s = SF_M^{-1}G_sF_M$$  (12)

where $G_s$ is the diagonal matrix with

$$\text{diag}(G_s) = g_s.$$  (13)

We denote $C_s$ the matrix given by,

$$C_s = F_M^{-1}G_sF_M.$$  (14)

$A_s$ is then given by

$$A_s = SC_s.$$  (15)

Consider $K = bL$, with $b$ an integer. Then if the output blocks are concatenated, the global $K$-point output signal $y_s$ obtained by OLS block filtering and its DFT $\overline{y}_s$ will be given by

$$y_s = H_s x \leftrightarrow \overline{y}_s = \overline{H}_s \overline{x}.$$  (16)

where $H_s$ is a $K \times K$ matrix containing $b$ copies of matrix $A_s$ as shown in Fig. 2. $\overline{H}_s$ is the $K \times K$ matrix defined by

$$\overline{H}_s = F_K H_s F_K^{-1}.$$  (17)

To evaluate the quality of the BDF, the criterion defined for designing the optimal filter is to minimize the quadratic error (mean square error) between the obtained output spectrum and the desired output spectrum:

$$e = E \left\{ \sum_{k=0}^{K-1} \left| \overline{y}_s(k) - \overline{y}_d(k) \right|^2 \right\}.$$  (18)

Using (8), (16), and the usual hypothesis (input signal modeled as white noise), minimizing $e$ in (18) is equivalent to minimizing the distortion error

$$e_s = \| \overline{H}_s - \overline{H}_d \|^2.$$  (19)

where $\| \cdot \|^2$ stands for the Frobenius norm. Since $F_K$ and $F_K^{-1}$ are unitary matrices, using the norm preserving properties gives that

$$e_s = \| H_s - H_d \|^2.$$.  (20)
C. Fast Algorithm for Optimal OLS Design

A fast algorithm for designing the optimal DBF for the OLS scheme shown above has been developed in [2]. The algorithm takes advantage of the circulant matrices properties [3] to reduce the calculation complexity.

Since $\overline{H}_d$ given in (9) is a diagonal matrix, $H_d$ given by (10) is a diagonal matrix and its first column is given by

$$H_d(:,0) = F_K^{-1} \text{diag}(\overline{H}_d).$$  \hspace{1cm} (21)

Since $H_d$ is a circulant matrix and $H_s$ has the particular structure shown in Fig. 2, then minimizing $e_s$ in (20) is equivalent to minimizing $e'_s$:

$$e'_s = ||A_s - A_1||^2$$  \hspace{1cm} (22)

where $A_1$ is the matrix extracted from $H_d$ according to the same structure shown in Fig. 2. Using (15), $e'_s$ can be described by

$$e'_s = ||SC_s - A_1||^2.$$  \hspace{1cm} (23)

As $G_s$ is a diagonal matrix, $C_s$ given by (14) is a circulant matrix. By using the pseudo-inverse and the relation between the rows of the matrix $C_s$ given by (4), we obtain

$$C_s(0,:)^T = \frac{1}{L} \sum_{i=0}^{L-1} \left( (P_M^{-1})^T \right)^{i+d} A_1(i,:)^T.$$  \hspace{1cm} (24)

This is equivalent to

$$C_s(0,j) = \frac{1}{L} \sum_{i=0}^{L-1} A_1(i, (i+j+d)_M).$$  \hspace{1cm} (25)

Finally, the elements of $g_s$ are obtained by

$$g_s = \text{diag}(G_s) = MF_M^{-1}(C_s(0,:))^T.$$  \hspace{1cm} (26)

IV. OVERLAP-ADD

A. Overlap-Add Scheme

The block digital filtering using the OLA scheme is illustrated in Fig. 3. The input signal is partitioned into nonoverlapping sequences $x'_i$ of length $L$. Then the following computations are performed.

- Each block is appended by $d = (M - L)/2$ zeros on each side to make it $M$-point block. Then, an $M$-point DFT is performed.
- The transformed block is multiplied term by term by an $M$-length vector $g_s$. In traditional approaches, this vector $g_s$ corresponds to the $M$-point DFT of the filter impulse response $h$ designed by one of the classical filter design methods.
- An inverse $M$-point DFT (IDFT) is performed.
- The resulting output signal is reconstructed by overlapping and adding the output blocks $y'_i$. The amount of overlapping in output blocks is $M - L = 2d$ samples.

In the OLA method, after computing each block output, we need to store $2d$ values of $y'_i$ and wait for the next output block data to add overlapped points. Thus, the number of the required real additions will be higher than that required in OLS method. However, the number $\mu$ of ripples is the same in both methods.

B. Optimal Overlap-Add Filter Design

By referring to Fig. 3, the input block $x'_i$ and the output block $y'_i$ are related by the relation:

$$y'_i = F_M^{-1} G_s F_M \begin{bmatrix} 0_{M \times 1} \\ x'_i \\ 0_{M \times 1} \end{bmatrix}$$  \hspace{1cm} (27)

where $G_s$ is the diagonal matrix with

$$\text{diag}(G_s) = g_s.$$  \hspace{1cm} (28)

This can be written as

$$y'_i = A_a x'_i$$  \hspace{1cm} (29)

where $A_a$ is the $M \times L$ matrix given by

$$A_a = F_M^{-1} G_s F_M S^T.$$  \hspace{1cm} (30)

$$S^T = \begin{bmatrix} 0_{d \times L} \\ I_L \\ 0_{d \times L} \end{bmatrix}$$ is the transpose of matrix $S$. By denoting

$$C_s = F_M^{-1} G_s F_M,$$

$A_a$ will be

$$A_a = C_s S^T.$$  \hspace{1cm} (32)

Consider $K = bL$, with $b$ an integer. Then, the global $K$-point output signal $y_a$ obtained by overlap-add and its DFT $\overline{y_a}$ will be

$$y_a = H_a x \iff \overline{y_a} = \overline{H_a x}.$$  \hspace{1cm} (33)
where $H_d$ is a $K \times K$ matrix containing $b$ copies of matrix $A_a$ as shown in Fig. 4. $\overline{H}_a$ is the $K \times K$ matrix defined by

$$\overline{H}_a = F_K H_a F_K^{-1}.$$  

(34)

To evaluate the quality of the BDF, we use the same criterion defined in the case of OLS: minimize the mean-square error. Using (8), (33), and the usual hypothesis (input signal modeled as white noise), (18) is equivalent to minimizing the distortion error

$$e_a = \| \overline{H}_a - \overline{H}_d \|^2.$$  

(35)

Using the norm preserving properties of the unitary matrices ($F_K$ and $F_K^{-1}$), (35) can be written as

$$e_a = \| H_a - H_d \|^2.$$  

(36)

C. Fast Algorithm for the OLA Scheme’s Optimal Design Approach

Just like in the previous case, since $H_d$ is a circulant matrix and given the particular structure of the $H_a$ matrix (see Fig. 4), then minimizing $e_a$ is equivalent to minimizing $e_a'$ given by

$$e_a' = \| A_a - A_2 \|^2$$  

(37)

where $A_2$ is extracted from $H_d$ according to the same structure shown in Fig. 4. Using (32), $e_a'$ will have the form

$$e_a' = \| C_a S^T - A_2 \|^2.$$  

(38)

As $C_a$ is a diagonal matrix, $C_a$ given by (31) is a circulant matrix. Then, by using the pseudo-inverse and the relations between the columns of the circulant matrix $C_a$ given by (5), we obtain

$$C_a(:,0) = \frac{1}{L} \sum_{i=0}^{L-1} ((P_M^{-1})^{i+d} A_2(:,i)).$$  

(39)

This is equivalent to

$$C_a(i,0) = \frac{1}{L} \sum_{j=0}^{L-1} A_2(i + j + d,M,j).$$  

(40)

Finally, the elements of $g_a$ are obtained by

$$g_a = \text{diag}(C_a) = F_M C_a(:,0).$$  

(41)

V. OPTIMAL TIME-IN Variant OLS AND OLA DESIGN

Linear time-invariant (LTI) filters are characterized by their coefficients which are fixed and do not change in time. In this case, the output signal $y(n)$ and the input signal $x(n)$ are related by the relation $y(n) = \sum_k h(k)x(n-k)$, where $h(n)$ is the impulse response of the filter. In the discrete frequency domain, the output spectrum $\overline{y}(k)$ and the input spectrum $\overline{x}(k)$ are related by $\overline{y}(k) = \overline{h}(k)\overline{x}(k)$, where $\overline{h}(k)$ is the frequency response of the filter.

Linear time-varying (LTV) filters are characterized by their coefficients which are not fixed and change over time. In this case, the output signal $y(n)$ and the input signal $x(n)$ are related by the relation $y(n) = \sum_k H(n,k)x(n-k)$, where $H(n,k)$ is the impulse response of the filter at time $n$. In the case where $H(n,k) = H(n + n_0,k)$, the system is called linear periodic time-varying (LPTV) with period $n_0$. In the discrete frequency domain, the relation between the output spectrum $\overline{y}(k)$ and the input spectrum $\overline{x}(k)$ can be described by the relation $\overline{y}(k) = \sum_l \overline{H}(k,l)\overline{x}(l)$. The output frequency component $\overline{y}(k)$ will depend not only on $\overline{x}(k)$ but also on all other components $\overline{x}(l)$, $l \neq k$. The term $\overline{H}(k,l)\overline{x}(l)$ is called LTI component and the terms $\overline{H}(k,l)\overline{x}(l)$ are aliased components. One of the most interesting tools to analyze the LPTV filter is the bifrequency map [7] which shows the time-invariant response of the filter as well as its aliasing components.

Due to the block structure, the OLS and OLA block filters are LPTV with period $L$ [1]. Our algorithm developed previously for designing the optimal OLS and OLA block filters takes into account both time-invariant and aliasing components. Our aim here is to derive the algorithms for designing the optimal time-invariant block filters. According to the (16) and (33), the relationships between the spectrums of the input and the output signals are respectively given by

$$\overline{y}_a(k) = \overline{H}_a(k,k)\overline{x}(k) + \sum_{l=0,l \neq k}^{K-1} \overline{H}_a(k,l)\overline{x}(l).$$  

(42)

$$\overline{y}_a(k) = \overline{H}_a(k,k)\overline{x}(k) + \sum_{l=0,l \neq k}^{K-1} \overline{H}_a(k,l)\overline{x}(l),$$  

(43)

$k = 0, 1, \ldots, K - 1$.

In the case of time-invariant filter, the aliasing components are zero which implies that $\overline{H}_a$ and $\overline{H}_a$ are diagonal matrices. Using the diagonalization property of circulant matrices, we can
infer that $H_a$ and $H_d$ are circulant matrices. According to Figs. 2 and 4, this will be the case when

$$A_s(i, j) = 0 \quad \text{for} \quad |j - i - d| > d \quad (44)$$
$$A_a(i, j) = 0 \quad \text{for} \quad |i - j - d| > d. \quad (45)$$

According to (15) and (32), the last two equations are true when

$$C_s(0, j) = 0 \quad \text{for} \quad d < j < M - d \quad (46)$$
$$C_a(i, 0) = 0 \quad \text{for} \quad d < i < M - d. \quad (47)$$

These restrictions must then be added to ensure time-invariance. Since the optimization of the BDF leads to independent coefficients $C_s(0, j)$ and $C_a(i, 0)$ which are calculated separately, the others coefficients of these two vectors will be the same as those obtained previously in the LPTV case by (25) and (40).

The two relations (46) and (47) are equivalent to the constraints obtained in the classical time-invariant OLS and OLA filter design. In fact, for causal filter, an order $p$ which is less than or equal to $M - L = 2d$ is required to ensure that the block OLS and OLA filtering is time-invariant, this means $h(n) = 0$, where $n > 2d$. However, in OLS and OLA filtering, the filters are not compelled to be causal. In Fig. 2, for example, the input and the output signal obtained in the OLS case are related by $y_b(n) = \sum_{\ell=0}^{M-1} A_s(n, \ell)x(n-d+\ell)$, $n = 0, 1, \ldots, K - 1$. The output signal $y_b(n)$ depends on past, present, and future inputs $x(n)$, so that the filtering is noncausal. In this case, the time-invariance constraint becomes $h(n) = 0$, where $|n| > d$. Using the periodicity property of the $M$-point DFT, the time-invariance is then given by $h(n) = 0$, where $d < n < M - d$.

VI. COMPARISON BETWEEN OPTIMAL OLA AND OPTIMAL OLS DESIGNS

In traditional design methods, $g_a$ and $g_s$ are obtained by the $M$-point DFT of the block filter impulse response. Hence, they are equal. Due to the time-invariance, this leads to the same filter frequency response and the same distortion error. Then, OLS and OLA show no difference. Our aim in this section is to show their difference when the filters are LPTV.

A. Relations Between $A_1$ and $A_2$, $C_s$ and $C_a$, $g_s$ and $g_a$

According to Figs. 2 and 4, we deduce that, for $b \geq 2$, the elements of $A_1$ and $A_2$ can be respectively given by

$$A_1(i, j) = H_d(i, j - d)K \quad (48)$$
$$(i : 0 \rightarrow L - 1, j : 0 \rightarrow M - 1).$$

$$A_2(i, j) = H_d(i - d)K, j \quad (49)$$
$$(i : 0 \rightarrow M - 1, j : 0 \rightarrow L - 1).$$

As noted before, $H_d$ given in (10) is a circulant matrix. Therefore, by using (3), we obtain

$$A_1(i, j) = A_2(M - 1 - j, L - 1 - i) \quad (50)$$
$$(i : 0 \rightarrow L - 1, j : 0 \rightarrow M - 1).$$

From (24) and (39), we can write

$$C_s(0, j) = \frac{1}{L} \sum_{\ell=0}^{L-1} A_1(i, \ell + j + d)M \quad (51)$$
$$C_a(i, 0) = \frac{1}{L} \sum_{\ell=0}^{L-1} A_2(i + j + d)M, j \quad (52)$$

$$(i, j : 0 \rightarrow M - 1).$$

Using (50), we deduce that

$$C_s(0, i) = C_a(M - 1 - i, 0). \quad (53)$$

Since the two matrices $C_s$ and $C_a$ are circulant, then we can infer that

$$C_s = C_a. \quad (54)$$

Consequently

$$g_s = g_a. \quad (55)$$

We note that, in the case where the diagonal elements of $H_d$ are real, $H_d$ will be a circulant Hermitian matrix ($H_d = H_d^* \quad (56)$). The obtained matrices $C_s$ and $C_a$ will be circulant Hermitian matrices, and then $g_s$ and $g_a$ will be real vectors. Furthermore, in the case where the diagonal elements of the $H_d$ matrix are real and symmetric, $H_d$ will have real elements, therefore it will be circular symmetric matrix ($H_d = H_d^\bullet \quad (57)$). The obtained matrices $C_s$ and $C_a$ will be circular symmetric matrices, and $g_s$ and $g_a$ will be real symmetric vectors: $(g_s(M - i) = g_s(i), g_a(M - i) = g_a(i), i : 0 \rightarrow M - 1)$.

B. Global Distortion Errors

Since $C_s$ and $C_a$ are equal, then $A_s$ and $A_a$ given respectively by the (15) and (32) will be related by

$$A_s(i, j) = A_a(M - 1 - j, L - 1 - i) \quad (56)$$
$$(i : 0 \rightarrow L - 1, j : 0 \rightarrow M - 1).$$

Therefore, due to their particular structures, $H_s$ and $H_a$ (Figs. 2 and 4) will be also related by

$$H_s(i, j) = H_s(K - 1 - j, K - 1 - i) \quad (57)$$
$$(i, j : 0 \rightarrow K - 1).$$

By developing the (20) and (36), we obtain

$$e_s = \|H_d - H_s\|^2 = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} \|H_d(i, j) - H_s(i, j)\|^2 \quad (58)$$
$$e_a = \|H_d - H_a\|^2 = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} \|H_d(i, j) - H_a(i, j)\|^2 \quad (59).$$

By denoting $i' = K - 1 - j$ and $j' = K - 1 - i$, and by using (57), we obtain

$$e_s = \sum_{j'=0}^{K-1} \sum_{i'=0}^{K-1} \|H_d(i', j') - H_a(i', j')\|^2 \quad (60).$$
Comparing (60) to (59), we can conclude that the global distortions associated to the optimal OLS and the optimal OLA are equal:

$$e_{\text{a}} = e_{\text{a}}.$$  \hspace{1cm} (61)

To enable more distortion comparison, we will consider the following separately:

— the aliasing distortion which is defined as the off-diagonal quadratic error between the desired frequency response matrix $\overline{P}_{\text{d}}$ and the obtained BDF frequency response matrix, $\overline{P}_{\text{s}}$ or $\overline{P}_{\text{a}}$;

— the time-invariant error which is the diagonal quadratic error.

C. Time-Invariant Components

The time-invariant distortion components for OLS and OLA are defined, respectively, by

$$\text{invar}_{\text{s}}(k) = \left| \overline{P}_{\text{s}}(k, k) - \overline{P}_{\text{d}}(k, k) \right|^2$$  \hspace{1cm} (62)

$$\text{invar}_{\text{a}}(k) = \left| \overline{P}_{\text{a}}(k, k) - \overline{P}_{\text{d}}(k, k) \right|^2.$$  \hspace{1cm} (63)

Equation (17) yields

$$\overline{P}_{\text{s}}(i, j) = \frac{1}{K} \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} F_{\text{s}}(i, m) H_{\text{s}}(m, n) F_{\text{s}}^{-1}(n, j).$$  \hspace{1cm} (64)

In a similar manner, we obtain

$$\overline{P}_{\text{a}}(i, j) = \frac{1}{K} \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} \alpha_{\text{a}}^{m+jn} H_{\text{a}}(m, n).$$  \hspace{1cm} (65)

By using (57) in (64), we obtain

$$\overline{P}_{\text{s}}(i, j) = \frac{1}{K} \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} \alpha_{\text{s}}^{m+jn} H_{\text{s}}(K-1-n, K-1-m).$$  \hspace{1cm} (66)

By denoting $m' = K - 1 - n$ and $n' = K - 1 - m$, we obtain

$$\overline{P}_{\text{s}}(i, j) = \frac{1}{K} \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} \alpha_{\text{s}}^{m+jn} H_{\text{s}}(m', n').$$  \hspace{1cm} (67)

By comparing (66) to (65), we deduce that

$$\overline{P}_{\text{s}}(i, j) = \alpha_{\text{s}}^{m+jn} \overline{P}_{\text{a}}(i, j).$$  \hspace{1cm} (68)

Then, for $j = i$, we infer that the diagonal elements will be equal:

$$\overline{P}_{\text{s}}(i, i) = \overline{P}_{\text{a}}(i, i).$$

Hence, the time-invariant filter components are equal for both OLS and OLA structures:

$$\text{invar}_{\text{s}}(k) = \text{invar}_{\text{a}}(k).$$  \hspace{1cm} (69)

D. Aliasing Components

The aliasing components for OLS and OLA are defined respectively by

$$\text{alias}_{\text{s}}(k) = \sum_{l=0, l \neq k}^{K-1} \left| \overline{P}_{\text{s}}(k, l) - \overline{P}_{\text{d}}(k, l) \right|^2$$  \hspace{1cm} (70)

$$\text{alias}_{\text{a}}(k) = \sum_{l=0, l \neq k}^{K-1} \left| \overline{P}_{\text{a}}(k, l) - \overline{P}_{\text{d}}(k, l) \right|^2.$$  \hspace{1cm} (71)

In the particular case when the block filters are time-invariant, aliasing components are zeros. In this case, OLS and OLA structures show no difference. However, when aliasing is present, the aliasing components are not the same and they show the different features.

Lemma 1: First, we can infer from (67) that the OLS and OLA aliasing components will be related by

$$\overline{P}_{\text{s}}(i, j) = \overline{P}_{\text{a}}(j, i).$$  \hspace{1cm} (72)

We note that a coefficient $\overline{P}_{\text{a}}(i, j) \neq j$, (idem $\overline{P}_{\text{a}}(i, j)$) represents the aliasing effect of the input frequency $j$ on the output frequency $i$.

Lemma 2: For both OLS and OLA, the aliasing impacting output frequency $i$ can be provided only by input frequencies $j$ such that

$$j - i = q \frac{K}{L}, \quad q \in \mathbb{Z}.$$  \hspace{1cm} (73)

This means that

$$\forall i, j, j - i \neq q \frac{K}{L}, q \in \mathbb{Z} \Rightarrow \overline{P}_{\text{a}}(i, j) = 0 = \overline{P}_{\text{a}}(i, j).$$  \hspace{1cm} (74)

Proof: We develop the calculation only for OLS. Calculation for OLA case can be developed in the same manner. According to (16), we have

$$\overline{P}_{\text{a}}(i) = \overline{P}_{\text{s}}(i, i) \alpha_{\text{a}}^i + \sum_{j=0, j \neq i}^{K-1} \overline{P}_{\text{s}}(i, j) \alpha_{\text{a}}^{j-i}$$  \hspace{1cm} (75)

where the first part is the time-invariant contribution and the second part, the aliasing. Due to the particular structure of the matrix $H_{\text{a}}$ (Fig. 2), by using (64), we obtain

$$\overline{P}_{\text{a}}(i, j) = \frac{1}{K} \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} \sum_{l=0}^{K-1} \alpha_{\text{a}}^{m+jn+U(j-i)l} A_{\text{a}}(m, n).$$  \hspace{1cm} (76)

where the first part is the time-invariant contribution and the second part, the aliasing. Due to the particular structure of the matrix $H_{\text{a}}$ (Fig. 2), by using (64), we obtain

$$\overline{P}_{\text{a}}(i, j) = \frac{1}{K} \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} \sum_{l=0}^{K-1} \left( \alpha_{\text{a}}^{m+jn+U(j-i)l} u_{\text{a}}^{l} A_{\text{a}}(m, n) \right)$$  \hspace{1cm} (76)

where $u = \alpha_{\text{a}}^{l(j-i)}$. Since $\alpha_{\text{a}}^{l(j-i)}$ is an $b$th root of the unity, then $\sum_{l=0}^{b-1} (\alpha_{\text{a}}^{l(j-i)})^l = 0$ except where $\alpha_{\text{a}}^{l(j-i)} = 1$ which is the case when $j - i = q\frac{K}{L}$ (we can also verify that by...
developing the sum of a geometric series). Then \( \overline{P}_a(i,j) \) is zero except when \( j - i \equiv q(K/L) \), which gives

\[
\overline{P}_a(i,j) = \frac{b}{K} \alpha_i^{jq} \sum_{m=0}^{L-1} \sum_{n=0}^{M-1} \alpha_K^{-im+jn} A_m(m,n), \quad (77)
\]

Lemma 3: In OLS, input frequencies \( j \) such that \( j = t(K)/\gcd(K,M) \), \( t \in \mathbb{Z} \), will not have aliasing effect on any output frequency \( i \). This means that

\[
(\forall i \neq j, j = t(K)/\gcd(K,M), t \in \mathbb{Z}) \Rightarrow \overline{P}_a(i,j) = 0. \quad (78)
\]

In OLA, aliasing on the output frequencies \( i \) such that \( i = t(K)/\gcd(K,M) \), \( t \in \mathbb{Z} \), will be equal to zero, this means that

\[
(\forall i \neq j, i = t(K)/\gcd(K,M), t \in \mathbb{Z}) \Rightarrow \overline{P}_a(i,j) = 0. \quad (79)
\]

Proof: We develop the calculation only for OLA. Calculation for OLS case can be developed in the same manner. Due to the particular structure of the matrix \( H_a \) (Fig. 4), we obtain from (65)

\[
\overline{P}_a(i,j) = \frac{\alpha_i^{jq} K}{L} \sum_{m=0}^{L-1} \sum_{n=0}^{M-1} (\alpha_K^{-im+jn} \sum_{\substack{l=0 \atop l \neq 0}}^{L-1} A_m(m,n)) \quad \text{(80)}
\]

where \( u \equiv \alpha_i^{jq} K \). As shown before in Lemma 2, \( \overline{P}_a(i,j) \) is zero except when \( j - i = t(K/L) \), which gives

\[
\overline{P}_a(i, i + qK/L) = \frac{\alpha_i^{jq} K}{L} \sum_{m=0}^{L-1} \sum_{n=0}^{M-1} (\alpha_K^{-im+jn} A_m(m,n))
\]

with \( \gamma = (b/K) \alpha_i^{jq} \). Since \( A_m(m,n+d) = c_{(n-m+\delta M)} \), we obtain

\[
\overline{P}_a(i, i + qK/L) = \frac{\alpha_i^{jq} K}{L} \sum_{m=0}^{L-1} \sum_{n=0}^{M-1} \alpha_K^{-im+jn} A_m(m,n)
\]

If \( \alpha_i^{-im} \equiv 1 \), this gives

\[
\overline{P}_a(i, i + qK/L) = \frac{\alpha_i^{jq} K}{L} \sum_{m=0}^{L-1} \sum_{n=0}^{M-1} \alpha_K^{-im+jn} A_m(m,n)
\]

Since \( \alpha_K^{qK/L} \equiv 1 \) is an \( L \)th root of unity, then \( \sum_{n=0}^{L-1} \alpha_K^{qK/L} = \sum_{n=0}^{L-1} 1 = L \), we can also verify that by develop the sum of a geometric series, therefore \( \overline{P}_a(i, i + qK/L) \) is zero when \( \alpha_i^{-im} \equiv 1 \). This is the case where

\[
iM = pK, \quad p \in \mathbb{Z}.
\]

By dividing by \( \lambda = \gcd(M,K) \), we obtain

\[
i = \frac{q}{K}M, \quad t \in \mathbb{Z}
\]

where \( \mu \) and \( \kappa \) are such \( \mu \equiv \mu \lambda \) and \( K \equiv \kappa \lambda \) (\( \mu \) and \( \kappa \) in \( \mathbb{Z} \)).

Since \( \mu \) and \( \kappa \) are relatively prime, then we obtain (Gauss Lemma)

\[
i = \frac{t}{K} \lambda = \frac{t}{\gcd(M,K)} \lambda.
\]

To conclude this section, we note that in LTI filtering, the aliasing is zero, then the OLS and OLA show no difference. However, when the filtering is LPTV, OLS and OLA show difference in the aliasing components. In this case, the aliasing components show the properties that we have described in this last subsection.

VII. RESULTS

We discuss experimental results obtained for \( M = 32 \), \( L = 24 \), and \( K = 504 \). Here, the desired spectral resolution is \( F_s/504 \) (\( F_s \) is the sampling frequency). These are relatively low values but we are interested in providing results that are easier to visualize. Equivalent results are obtained for practical applications in which large values of block sizes (\( M, L \)) and \( K \) are required.

Fig. 5 shows the desired frequency response \( f(k)(k : 0 \rightarrow 5(K)) \). The horizontal axis is the frequency indexes (\( k \) represents the normalized frequency \( \omega = 2\pi k/K \)). \( f(k) \) is equal to 0 between indexes 102 and 402 and 1 elsewhere. In Table I, we compare the time-invariant BDF designs provided by our optimal approach to that provided by some traditional approaches [4], [5]. In least square and Hamming window methods, the order of the filter is equal to \( M - L = 2d = 8 \). We see that our optimal LTI filter is the same as that obtained by the least square method while our optimal LPTV design approach provides lower distortion by tolerating a slight aliasing which provides a considerable gain on the overall quality of the filter.

For illustration, Figs. 6, 7, and 8 show the matrices \( \overline{P}_a \) which represent the 3-D frequency responses: the desired, the optimal OLS and the optimal OLA. \( \overline{P}_a \) and \( \overline{P}_a \) are nondiagonal, hence, aliasing is present. The represented values correspond to the quadratic norm \( |\overline{P}_a(i,j)|^2 \).

Figs. 9, 10, and 11 show the difference between the optimal OLS and the optimal OLA design approaches. The horizontal
axis designates the frequency index and the vertical axis designates the errors in logarithmic scale (in decibels). As proved above, the components of the time-invariant errors are the same. The values represented in Fig. 9 correspond to \( \text{invar}_a(k) = \text{invar}_a(k) \) defined in (62) and (63). We can note that there are peaks of time-invariant errors near the limits of the filter bandpass. The values represented in Figs. 10 and 11 are, respectively, the values of aliasing defined in (70) and (71). In both implementations, there are peaks of aliasing near the limits of the filter bandpass. That means that a large part of the distortion error is concentrated at frequencies that may be reserved for guard intervals. In OLA implementation, we can see and verify, as developed in the previous section, that aliasing is zero for frequency index \( k \) such that \( k = b(K/\gcd(K, M)) = 63b, b \in \mathbb{Z} \).

Fig. 12 shows matrix \( \overline{H}_a \) obtained in optimal OLS design. The values is represented by a gray value that is decreasing function of \( \sqrt{|t|} \)\( \overline{H}_a(i, j) \) (1 = black, 0 = white). We can see and verify, as developed in the previous section, that \( \overline{H}_a(i, j) = 0 \) when \( j - i \neq q(K/L) = 21q, q \in \mathbb{Z} \), and that \( (\forall i \neq j, j = t(K/\gcd(K, M)) = 63t, t \in \mathbb{Z} \Rightarrow \overline{H}_a(i, j) = 0 \).

VIII. Conclusion

In this paper, we have developed a simple and efficient matrix-oriented method to optimize the quadratic error in block digital filters (BDF) by analyzing overlap save (OLS) and overlap add (OLA) implementations. This approach improves the BDF frequency response and is based on elementary matrix
computations; hence, it can be simply implemented in block digital filtering applications. In addition to the general approach which leads to a time-variant system, we have developed the algorithm for designing the optimal time-invariant block filtering. We have shown that the difference between OLS and OLA implementations will be in the aliasing components when aliasing is not null. Finally, the usefulness of the optimal approach has been validated by comparing it to traditional BDF design approaches.

REFERENCES


Ali Daher was born in 1983. He received the M.S. degree from the University of Technology of Compiègne, France, in 2006, and the Ph.D. degree from University of Brest, France, in 2009. He is currently an instructor and a researcher at the Lab-STIC Laboratory (UMR CNRS 3192), University of Brest, France. His research interests include signal processing for digital communications, number theoretic transform, and its applications in fast digital filtering algorithms.

El Houssaïn Baghious was born in Morocco in 1955. He received the Ph.D. degree from the University of Brest, France, in 1991. Since 1992, he has been an Associate Professor and a researcher at the Lab-STICC Laboratory (UMR CNRS 3192), University of Brest, France. His research interests are in the areas of number theoretic transform and its applications to signal processing for digital communications.

Gilles Burel (M’00–SM’08) was born in 1964. He received the M.Sc. degree from Ecole Supérieure d’Électricité, Gif Sur Yvette, France, in 1988 and the Ph.D. degree and Habilitation to Supervise Research degree from the University of Brest, France, in 1991 and 1996, respectively. From 1988 to 1997 he was with Thomson CSF, then Thomson Multimedia, Rennes, France, where he worked on image processing and pattern recognition applications as Project Manager. Since 1997, he has been Professor of Digital Communications, Image and Signal Processing at the University of Brest. He is Associate Director of the Laboratory for Science and Technologies of Information, Communication and Knowledge (Lab-STIC—UMR CNRS 3192). He is author or coauthor of 19 patents, one book, and 140 scientific papers. His present research interests are in signal processing for digital communications, MIMO systems, and interception of communications.

Emanuel Radoi was born in 1968. He graduated in radar systems at the Military Technical Academy of Bucharest in 1992. He received the M.S. degree in electronic engineering and the Ph.D. degree in signal processing, both from the University of Brest, in 1997 and 1999, respectively. He coauthored seven books and more than 75 journal and conference papers. He is currently Professor of Signal Processing in the Electronics Department of the University of Brest. His main research interests include superresolution methods, time-frequency analysis, fast digital filtering algorithms, and UWB signal processing.